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Using two intervals we get the value of π from $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$ by Simpson's rule, 3.14093, with the error .00066; by Rule 1, 3.14178, with the error .00019; and by Rule 3, 3.14161, with the error .00002.

SOME METRICAL PROPERTIES OF THE PENTAHEDROID IN A SPACE OF FOUR DIMENSIONS.

By M. H. SZNYTER, University of California.

The purpose of this paper is to establish for the pentahedroid in a four-dimensional space, theorems similar to those found in ordinary solid geometry, dealing with the tetrahedron. The terms line, point and plane are used with the same significance as in three-dimensional geometry. By hyperplane we shall mean that three-dimensional element which consists of any four points not points of one plane, all points collinear with any two of them or with any two obtained by this process.[1] We shall first develop the simpler theorems, then we shall consider the pentahedroid with its tangent hyperspheres.

THEOREM 1.[2] *Let* $A_1A_2A_3A_4A_5$ *be a pentahedroid cut by a hyperplane* α *in such a way that the edge* A_1A_2 *lies on one side of* α *and the face* $A_3A_4A_5$ *on the other side. Then the following cases will appear.*

1. *If* α *is parallel to the line* A_1A_2 *and to the plane* $A_3A_4A_5$, *the section will be a prism.*
2. *If* α *is parallel to the line* A_1A_2 *but not to the plane* $A_3A_4A_5$, *the section will be a truncated prism.*
3. *If* α *is parallel to the plane* $A_3A_4A_5$ *but not to the line* A_1A_2, *the section will be a frustrum of a pyramid.*
4. *If* α *is parallel neither to the plane* $A_3A_4A_5$ *nor to the line* A_1A_2, *the section will be a truncated pyramid.*

For the hyperplane α cuts the tetrahedrons $A_2A_3A_4A_5$ and $A_1A_3A_4A_5$ in planes which cut a triangle from each $A_3'A_4'A_5'$ and $A_3''A_4''A_5''$. Since α passes between the edges A_1A_2 and the edges A_3A_4, A_4A_5 and A_3A_5, it must cut the other three tetrahedrons in planes which cut quadrilaterals from them. The section cut out from the pentahedroid will be a three-dimensional figure having for its boundaries two triangular faces and three quadrilateral faces. In cases 1 and 2 the lateral edges of the section are parallel and the section is prismatic. In case 1 the triangular faces are parallel while in case 2 they are not. Hence the former gives a prism while the latter gives a truncated prism. In cases 3 and 4, the lateral edges of the section will meet, if produced, at the point P which is the point of intersection of A_1A_2 with α, and the section is pyramidal. When α is parallel to plane $A_3A_4A_5$, the triangles $A_3'A_4'A_5'$ and $A_3''A_4''A_5''$ are parallel;

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[1] MANNING: *Geometry of Four Dimensions*, p. 24.

[2] *Ibid.*, p. 228.

the section is thus a frustrum of a pyramid, as in case 3. When α is not parallel to the plane $A_3A_4A_5$, the planes of the triangles $A_3'A_4'A_5'$, $A_3''A_4''A_5''$ meet if produced in the line g, the intersection of α and $A_3A_4A_5$. This section is therefore a truncated pyramid.

THEOREM 2. *The lateral volumes of two similar pentahedroids are to each other as the cubes on their homologous sides.*

THEOREM 3. *Pentahedroids are to each other as the products of their hyperplane bases by their altitudes; hence pentahedroids which have equivalent bases are to each other as their altitudes and pentahedroids which have equivalent altitudes are to each other as their bases.*

THEOREM 4. *Pentahedroids which have a hyperspace angle of one equal to a hyperspace angle of the other are to each other as the products of the edges of the equal hyperspace angles.*

From theorems 3 and 4 we get at once,

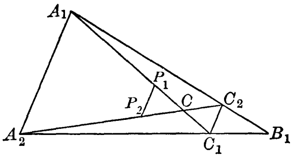
THEOREM 5. *Similar pentahedroids are to each other as the fourth powers of their homologous edges.*

THEOREM 6. *The lines drawn from the five vertices of a pentahedroid to the centers of gravity of the opposite cells cut each other in two segments in the ratio of 4 : 1.*

Consider the pentahedroid $A_1A_2A_3A_4A_5$ with C_k the center of gravity of the cell opposed to the vertex A_k. Since we know that the segments A_1C_1, A_2C_2, $\cdots$ are concurrent at the center of gravity of the pentahedroid we may denote this point by C. The following relations shall hold:

$$A_1C : CC_1 = 4 : 1, \quad A_2C : CC_2 = 4 : 1, \quad \cdots.$$

For upon the segment A_1C_1 consider a point P_1 on the same side of C as A_1 and such that CP_1 equals $\frac{1}{4}CA_1$. Consider four other points P_k similarly situated upon the other four segments A_kC_k.



Then the segments A_1C_2 and A_2C_1 meet at a point B_1 which is the center of gravity of the face $A_3A_4A_5$. For C_2 is the center of gravity of the cell $A_1A_3A_4A_5$ and hence A_1C_2 passes through the center of gravity of the face $A_3A_4A_5$. Likewise C_1 is the center of gravity of the cell $A_2A_3A_4A_5$ and hence A_2C_1 also passes through the center of gravity of the face $A_3A_4A_5$. We now have two triangles $A_1A_2B_1$ and A_1A_2C in the plane determined by A_1B_1 and A_2B_1.

The segment C_1C_2 is parallel to A_1A_2 and is equal to $\frac{1}{4}A_1A_2$, for

$$C_1B_1 = \tfrac{1}{4}A_2B_1 \quad \text{and} \quad C_2B_1 = \tfrac{1}{4}A_1B_1.$$

Similarly the segment P_1P_2 is parallel to A_1A_2 and is equal to $\frac{1}{4}A_1A_2$. Thus

P_1P_2 and C_1C_2 determine a parallelogram, from which relation we obtain at once,

$$CC_1 = P_1C = \tfrac{1}{4}A_1C \quad \text{and} \quad CC_2 = P_2C = \tfrac{1}{4}A_2C$$

or $A_1C : CC_1 = 4 : 1$, etc., which proves our theorem.

PROBLEM 1. *To find the radius of the inscribed and of the circumscribed hyperspheres of any regular pentahedroid in terms of its edge.*

Consider the regular pentahedroid $A_1A_2A_3A_4A_5$ with an edge equal to a certain finite magnitude K. Its ten edges, ten faces, dihedral angles and cells are all equal among each other for we know that the regular pentahedroid is congruent to itself in sixty different ways. It is first necessary to determine the radii of circles inscribed in and circumscribed about the plane faces and also the radii of the spheres inscribed and circumscribed about the five cells of the pentahedroid.

Consider one of the regular faces, $A_1A_2A_3$. By plane geometry the altitude of this triangle is

$$a = \sqrt{K^2 - \tfrac{1}{4}K^2} = \frac{\sqrt{3}}{2}K.$$

Since the three altitudes intersect at the center of gravity of this triangle, the radius of the inscribed circle is one third of the altitude and the radius of the circumscribed circle is two thirds of the altitude. Thus

$$r_i = \frac{1}{3}\frac{\sqrt{3}}{2}K = \frac{\sqrt{3}}{6}K, \quad r_c = \frac{2}{3}\frac{\sqrt{3}}{2}K = \frac{\sqrt{3}}{3}K.$$

Consider one of the regular cells as $A_1A_2A_3A_4$. Its altitude A is measured by the length of one leg of a right-angled triangle whose hypotenuse is the edge K and whose other leg is the radius of the circle circumscribed about the regular face $A_1A_2A_3$. Hence

$$A = \sqrt{K^2 - \frac{3K^2}{9}} = \frac{\sqrt{6}}{3}K.$$

The altitudes of the tetrahedron meet at its center of gravity and hence the radius of the inscribed sphere is one fourth the length of the altitude of the cell and the radius of the circumscribed sphere is three fourths the altitude. Therefore,

$$R_i = \frac{1}{4}\frac{\sqrt{6}K}{3} = \frac{\sqrt{6}}{12}K, \qquad R_c = \frac{3}{4}\frac{\sqrt{6}}{3}K = \frac{\sqrt{6}}{4}K.$$

In the pentahedroid $A_1A_2A_3A_4A_5$, the altitudes are concurrent at the center of gravity of the pentahedroid, therefore the radius of the inscribed hypersphere is equal to one fifth of an altitude and the radius of the circumscribed hypersphere is equal to four fifths of an altitude. Let α represent any altitude of the pentahedroid and ρ_i, ρ_c the radii of the inscribed and circumscribed hyperspheres respectively. As α is measured by the length of one leg of a right-angled triangle whose hypotenuse is the edge K and whose other leg is the radius R_c of the sphere circumscribed about one of its cells, therefore,

$$\alpha = \sqrt{K^2 - \frac{6}{16}K^2} = \frac{\sqrt{10}}{4}K$$

and

$$\rho_1 = \frac{1}{5}\alpha = \frac{\sqrt{10}}{20}K,$$

radius of the inscribed hypersphere

$$\rho_c = \frac{4}{5}\alpha = \frac{\sqrt{10}}{5}K,$$

radius of the circumscribed hypersphere.

Turning now to the second part of our problem, namely the tangent hyperspheres of the pentahedroid, it is first necessary to establish a system of coördinates for any given point of hyperspace with reference to a fixed pentahedroid. This system may be called pentahedroidal and will be based upon the distances of any point in hyperspace from the five cells of the pentahedroid. It is necessary to find a relation between these distances. They will then be sufficient to determine a single point in hyperspace.

Consider the pentahedroid $A_1A_2A_3A_4A_5$. Let V_k be the volume of the cell opposite the summit A_k and x_k the distance of any point P from the hyperplane V_k. Then the necessary and sufficient condition that any five quantities x_1, x_2, x_3, x_4, x_5 represent the coördinates of the point P with respect to the pentahedroid is that they satisfy the following equation,

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H, \tag{1}$$

where H is the hypervolume of the pentahedroid. The five hyperplanes of the pentahedroid divide hyperspace into thirty-one regions, as follows:

1. The interior $A_1A_2A_3A_4A_5$ of the pentahedroid.
2. The region α having only the vertex A_1 in common with the pentahedroid and lying on the same side of $A_2A_3A_4A_5$ that A_1 lies. There are five regions of this sort, one opposed to each vertex of the pentahedroid.
3. The region β bounded by three hyperplanes having the edge A_2A_3 in common and two hyperplanes having each only one point in common with the edge A_2A_3. There are ten regions of this sort, one opposite each edge of the pentahedroid.
4. The region γ bounded by two hyperplanes having the face $A_2A_3A_4$ in common and by three hyperplanes having each only a line in common with the face $A_2A_3A_4$. There are ten regions of this sort, one relative to each face.
5. The region δ formed by removing the pentahedroid from the interior of one of its tetrahedroidal angles. There are five such regions formed by taking each of the points A_1, A_2, $\cdots$ in turn.

Then for any point lying in any of the regions so determined, the perpendicular distances from it to the five hyperplanes of the pentahedroid shall satisfy equation (1).

Consider P any point within the pentahedroid and x_1, x_2, x_3, x_4, x_5 the distances from this point to the five cells of the pentahedroid which is now divided into five smaller pentahedroids formed by the five cells taken in turn with the point P as a common vertex. The hypervolume of the original pentahedroid is equal to the sum of the hypervolumes of the five smaller pentahedroids. Since the hypervolume of any pentahedroid is equal to one fourth the product of its hyperplane base by its altitude[1] we have at once,

$$4H = V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5.$$

If the point P is within the region α, the pentahedroid $A_1A_2A_3A_4A_5$ is equal to the pentahedroid $PA_2A_3A_4A_5$ minus the sum of the other four pentahedroids determined by P with the other four cells of the original pentahedroid. The altitudes of the five pentahedroids having P for a vertex are x_1, $-x_2$, $-x_3$, $-x_4$, $-x_5$.[2] Hence

$$4H = V_1x_1 - (-V_2x_2) - (-V_3x_3) - (-V_4x_4) - (-V_5x_5)$$

or

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

When P is within the region β, the five altitudes of the five pentahedroids determined by P are $-x_1$, x_2, x_3, $-x_4$ and $-x_5$. Thus

$$4H = V_2x_2 + V_3x_3 - (-V_4x_4) - (-V_5x_5) - (-V_1x_1),$$

or

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

For P in the region γ the altitudes of the five pentahedroids determined by it are $-x_1$, x_2, x_3, x_4, $-x_5$ and for ρ in the region δ the altitudes are $-x_1$, x_2, x_3, x_4, and x_5. The relations between the hypervolumes become

$$4H = V_2x_2 + V_3x_3 + V_4x_4 - (-V_1x_1) - (-V_5x_5) \text{ for } \gamma,$$

$$4H = V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 - (-V_1x_1) \qquad \text{for } \delta,$$

whence

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

From the above consideration it is evident that if a point P lies within any one of the regions determined by the five hyperplanes of a pentahedroid, its distances to the five hyperplanes satisfy a certain relation which is expressed by equation (1). To complete the establishment of the system it is necessary to show that if any five quantities x_1, x_2, x_3, x_4 and x_5 satisfy equation (1) they determine one and only one point in hyperspace.

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[1] MANNING: *Geometry of Four Dimensions*, pp. 277–278.

[2] Any distance x_k shall be positive or negative depending on whether it is on the same side or opposite side of the hyperplane V_k that the vertex A_k is.

Consider the hyperplanes P_1, P_2, P_3, P_4 and P_5 at the distances x_1, x_2, x_3, x_4 and x_5 from the hyperplanes V_1, V_2, V_3, V_4 and V_5 and such that P_k is parallel to V_k. Any four hyperplanes will meet at a point in hyperspace.

Suppose P_1, P_2, P_3 and P_4 meet at Q and let g_1, g_2, g_3, g_4 and g_5 be the coördinates of the point φ with respect to the pentahedroid. From (1) we have

$$V_1g_1 + V_2g_2 + V_3g_3 + V_4g_4 + V_5g_5 = 4H.$$

From the construction of the hyperplanes P_1, P_2, P_3, P_4, we have

$$g_1 = x_1, \qquad g_2 = x_2, \qquad g_3 = x_3, \qquad g_4 = x_4.$$

Therefore

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5g_5 = 4H.$$

Also

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

Therefore $g_5 = x_5$ and Q is a single point of hyperspace.

THEOREM 7. *There are at most sixteen hyperspheres tangent to the five cells of a pentahedroid.*

1. If there be a hypersphere inscribed within the pentahedroid, the coördinates of the center C, besides being all equal, must satisfy equation (1). If R be the radius of the hypersphere, then

$$x_1 = x_2 = x_3 = x_4 = x_5 = R,$$

and hence

$$RV_1 + RV_2 + RV_3 + RV_4 + RV_5 = 4H,$$

$$R(V_1 + V_2 + V_3 + V_4 + V_5) = 4H,$$

$$R = \frac{4H}{V_1 + V_2 + V_3 + V_4 + V_5}. \tag{2}$$

Since H, V_1, V_2, V_3, V_4, V_5 are all positive, (2) gives a single positive value of R, hence there is a single hypersphere inscribed in the pentahedroid, tangent to its five cells.

2. If there exists a hypersphere tangent to the cells with its center within the region α, the coördinates of the center will satisfy the following,

$$x_1 = -x_2 = -x_3 = -x_4 = -x_5 = R$$

and

$$RV_1 - RV_2 - RV_3 - RV_4 - RV_5 = 4H,$$

whence

$$R = \frac{4H}{V_1 - (V_2 + V_3 + V_4 + V_5)}. \tag{3}$$

As any four cells of the pentahedroid are together greater than the fifth, the right-hand member of (3) is negative, and since a negative value of R is inadmissible, there is no hypersphere tangent to the cells with its center in the region α. Therefore there are no tangent hyperspheres in the other four regions of the type α.

3. If there be tangent hyperspheres in the regions β and γ, the following must hold:

$$-x_1 = x_2 = x_3 = -x_4 = -x_5 = R,$$

$$-x_1 = x_2 = x_3 = x_4 = -x_5 = R,$$

and

$$R = \frac{4H}{V_2 + V_3 - (V_1 + V_4 + V_5)} \text{ in } \beta, \tag{4}$$

$$R = \frac{4H}{V_2 + V_3 + V_4 - (V_1 + V_5)} \text{ in } \gamma. \tag{5}$$

The value of R is now dependent on whether the sum of two cells of a pentahedroid is less than, equal to or greater than the sum of the other three cells. Thus if $V_2 + V_3 - (V_1 + V_4 + V_5)$ is negative in (4) then $V_1 + V_4 + V_5 - (V_2 + V_3)$ in (5) will be positive giving a positive R in γ for a negative value of R in β. Thus the number of tangent hyperspheres in the regions β and γ together will always be ten unless some of these expressions happen to be zero.

4. Finally when the center of the tangent hypersphere lies within the region δ,

$$-x_1 = x_2 = x_3 = x_4 = x_5 = R,$$

$$R = \frac{4H}{V_2 + V_3 + V_4 + V_5 - V_1}. \tag{6}$$

As the sum of any four cells of a pentahedroid is greater than the fifth, (6) gives a single positive value for R and hence there is one hypersphere tangent to the cells with its center in the region δ. There are five such hyperspheres, one in each of the five regions of the type δ.

Thus we have one tangent hypersphere within the pentahedroid, ten within the regions β and γ taken together and five in the regions δ, making the number of tangent hyperspheres equal in general to sixteen.

BOOK REVIEWS.

Send all communications to W. H. Bussey, University of Minnesota.

Plane and Solid Geometry. By Webster Wells and Walter W. Hart. D. C. Heath and Company, Boston, 1916. viii + 467 pages.

It is very important that the introduction to a course in plane geometry should be well done. With inexperienced teachers, at least, the introduction to the course will usually be the one that is given in the text. The introductory chapter of a geometry may be expected to make a clear statement of certain definitions and assumptions and to illustrate them; to offer some suggestions that will tend to make the students feel a need for geometry; and to show the need of formal proofs and to give an introduction to them. In the present text